Integrality Gap for the Knapsack Problem

Note: These notes are still somewhat rough.

Suppose we have n items of unit size and we want to pack as many as possible in a knapsack of size r. Then, we can pack at most $\lfloor r \rfloor$ number of items in the knapsack. In this lecture, we show that this seemingly simple reasoning that one can't pack a "fractional" item is not captured by low-degree SoS algorithm. Specifically, we'll prove Grigoriev's theorem (Grigoriev [2001]):

1. Theorem (SoS Hardness for Knapsack). *For any* $r \leq n/2$, *there is a pseudodistribution of degree* $\Omega(r)$ *supported on points* $x \in \{0,1\}^n$ *such that*

$$\sum_{i=1}^{n} x_i = r. \tag{1}$$

Observe that for non-integral r the above yields an integrality gap for the special case of knapsack presented above of degree $\Omega(r)$.

Grigoriev's original proof of this result was a elegant argument that analyzed a natural, maximally symmetric pseudodistribution that we will momentarily describe. This argument essentially invented several basic results about the spectrum of matrices from the *Johnson Scheme* studied in algebraic combinatorics. We highly recommend reading Grigoriev's original proof - in the lecture however, we will rely on standard results from the theory of association schemes and get a shorter proof. This argument was first presented by Meka and Wigderson (Meka and Wigderson [2013]) in a work that attempted to show the first SoS lower bound for the Planted Clique problem.

The Pseudodistribution

Fix $d = \Theta(r)$ to be chosen later. The idea for constructing degree d pseudodistribution is very natural - the constraint $\sum_i x_i = r$ is symmetric in r. Thus, if we had an arbitrary degree $2d$ pseudodistribution, we could average it over all permutations σ of $[n]$ and obtain another degree d pseudodistribution over the hypercube that is symmetric w.r.t. permutations of $[n]$ and satisfies Eq. (1).

Thus, for any symmetric pseudodistribution μ, there exists a function f such that for every multilinear monomial $x_S = \Pi_{i \in S} x_i$, $\tilde{\mathbb{E}}[x_S] = f(|S|)$. It turns out that there's no choice in f either.

First, $\sum_{i=1}^{n} \tilde{\mathbb{E}}_{\mu}[x_i] = nf(1)$. On the other hand, since μ is supported on x satisfying Eq. (1), $\sum_{i=1}^{n} \tilde{\mathbb{E}}[x_i] = r$. Thus, $f(1) = r/n$ for every i.

$$rf(1) = r\,\tilde{\mathbb{E}}[x_i] = \tilde{\mathbb{E}}[x_i(\sum_{j=1}^{n} x_j)] = \tilde{\mathbb{E}}[x_i] + (n-1)\,\tilde{\mathbb{E}}[x_i x_j] = f(1) + (n-1)f(2),$$

(2)

which implies that $f(2) = (r-1)/(n-1)$. One can repeat this argument to obtain that

$$\tilde{\mathbb{E}}[x_S] = f(|S|) = \frac{\binom{r}{|S|}}{\binom{n}{|S|}},$$

(3)

for every S such that $|S| \le 2d$.

It is easy to check that $\tilde{\mathbb{E}}$ constructed here satisfies the constraint Eq. (1). Thus, as usual, we are left with showing the positivity of $\tilde{\mathbb{E}}$.

Reduction to Positivity over Squared Homogenous Polynomials

Because of the linear equality constraint Eq. (1), it turns out that it is enough to prove positivity of $\tilde{\mathbb{E}}[p^2]$ for any homogenous degree d polynomial.

2. Lemma (Positivity over Squared Homogenous Polyomials is Enough). *Suppose M be any linear operator on degree d polynomials that is consistent with the constraints $x_i^2 = x_i$ for every $i \in [n]$ and $\sum_{i=1}^{n} x_i - r = 0$. Suppose $M(p^2) \ge 0$ for every degree d homogenous polynomial p. Then, $M(q^2) \ge 0$ for any degree d polynomial q.*

Proof Sketch. The idea is that any polynomial p of degree d can be written as

$$p_1 + \sum_{i=1}^{n}(x_i^2 - x_i)p_i + q(\sum_{i=1}^{n} x_i - r) \tag{4}$$

for a homogenous degree d polynomial p_1, degree at most $d-2$ polynomials p_i and degree at most $d-1$ polynomial q. This can be shown by polynomial division, for example.

Once we have the above representation, squaring the right hand side of Eq. (4) yields p_1^2 plus terms that have either $(x_i^2 - x_i)$ or $(\sum_{i=1}^{n} x_i - r)$ as a factor. Applying M to the RHS, using linearity and the fact that M satisfies both Boolean and the knapsack constraints thus yields that $M(p^2) = M(p_1^2) \ge 0$. $\square$

We can now go to the matrix view of things in order to show positivity of $\tilde{\mathbb{E}}$ on homogenous degree d polynomials - we have seen this argument several times in the lectures before.

3. Exercise (Moment Matrix). Let $\mathcal{M} \in \mathbb{R}^{\binom{n}{d} \times \binom{n}{d}}$ be defined by $\mathcal{M}(S,T) = \tilde{\mathbb{E}}[x_{S \cup T}] = f(2d - |S \cap T|)$ for any $S, T \in \binom{n}{d}$. Show that $\tilde{\mathbb{E}}[p^2] \geq 0$ for every homogenous degree d polynomial if and only if $\mathcal{M} \succeq 0$.

Note that $\mathcal{M}$ is the $\binom{n}{d} \times \binom{n}{d}$ dimensional principal sub-matrix of the usual moment-matrix of $\tilde{\mathbb{E}}$.

Johnson Scheme

The discussion here is based on (Meka and Wigderson [2013]) which in turn is based on (Godsil [1993]). Association schemes are well-studied objects in Algebraic Combinatorics. For our purposes, we can think of them as a *commutative algebra* of square matrices - i.e. adding or multiplying any two matrices from the set yields another matrix in the set and for any two matrices A, B in the set, $AB = BA$. We are interested in one such well-studied scheme.

4. **Definition (Johnson Scheme).** Let $d < n/2 \in \mathbb{N}$ be parameters. The *Johnson Scheme* $\mathcal{J}_{n,d}$ of order d on $[n]$ is the linear subspace of all matrices J in $\mathbb{R}^{\binom{n}{d} \times \binom{n}{d}}$ that are *set symmetric* i.e. $J(I,J) = h(|I \cap J|)$ for any $I, J \in \binom{n}{d}$. In other words, any entry of any matrix in the subspace depends only on the size of the intersection of the row and column index sets.

We now define two basis for $\mathcal{J}_{n,d}$.

5. **Definition (D Basis).** For $0 \leq \ell \leq d \leq n$, let $D_\ell \in \mathbb{R}^{\binom{n}{d} \times \binom{n}{d}}$ be the matrix defined by

$$D_\ell(I,J) = \begin{cases} 1 & \text{if } |I \cap J| = \ell \\ 0 & \text{otherwise..} \end{cases} \tag{5}$$

D_0 is then the well-studied Set Disjointness matrix from communication complexity. It is easy to check that D_ℓ for $\ell \leq d$ span $\mathcal{J}_{n,d}$. Further, it's also easy to verify that the D_ℓ's commute with each other - and thus every pair of matrices in $\mathcal{J}_{n,d}$ commute establishing that $\mathcal{J}_{n,d}$ is indeed a commutative algebra of matrices.

For the purposes of proving PSDness, another basis of $\mathcal{J}_{n,d}$ is very useful - the P-Basis.

6. **Definition (P Basis).** For $0 \leq t \leq d$, let $P_t \in \mathbb{R}^{\binom{n}{d} \times \binom{n}{d}}$ be defined by

$$P_t(I,J) = \binom{|I \cap J|}{t}, \tag{6}$$

where it's understood that if $|I \cap J| = 0$, then $P_t(I, J) = 0$.

There's an equivalent definition of P_ts that's helpful in calculations.

7. Exercise (Alternate Definition of P-Basis). Let R_T be the rank 1 matrix defined by $R_T(I, J) = 1(I \supseteq T)1(J \supseteq T)$. Show that $P_t(I, J) = \sum_{T \subseteq [n], |T| = t} R_T$.

The above exercise can be used to obtain explicit basis change coefficients between the D and the P basis.

8. Exercise (Change between P and D Basis). Show that

1. For $0 \leq t \leq d$, we have $P_t = \sum_{\ell=t}^{d} \binom{\ell}{t} D_\ell$.

2. For $0 \leq \ell \leq d$, we have $D_\ell = \sum_{t \geq \ell} (-1)^{t-\ell} \binom{t}{\ell} P_t$.

The main results from the theory of association schemes of interest to us is the characterization of eigenspaces and eigenvalues of the matrices in $\mathcal{J}_{n,d}$. This is done using the fact that there's a natural action (the relabeling action on elements of $[n]$) of S_n that commutes with every matrix in $\mathcal{J}_{n,d}$ (because of set-symmetry of $\mathcal{J}_{n,d}$-matrices). This implies that matrices in $\mathcal{J}_{n,d}$ must share eigenspaces with those corresponding to the action of the symmetric group S_n. The latter are well understood as *representations* of S_n and one can use this understanding to arrive at an explicit description of eigenvalues and eigenspaces of matrices in $\mathcal{J}_{n,d}$.

For our purposes, we will just state the results required for us.

9. Lemma (Eigenspaces of Johnson Scheme). *For $P_t = P_{t,n,d}$ defined as above, there exist pairwise orthogonal subspaces $V_0, V_1, V_2, \ldots, V_d$ such that*

1. *$V_0, V_1, V_2, \ldots, V_d$ are eigenspaces for P_t for every $0 \leq t \leq r$ and consequently for every matrix in $\mathcal{J}_{n,d}$.*

2. *$dim(V_j) = \binom{n}{j} - \binom{n}{j-1}$.*

3. *For any matrix $J \in \mathcal{J}_{n,d}$, let $\lambda_i(J)$ denote the eigenvalue of J on the subspace V_i. Then,*

$$\lambda_i(P_t) = \begin{cases} \binom{n-t-i}{d-t} \cdot \binom{d-i}{t-i} & \text{if } i \leq t \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

The above lemma helps us estimate the eigenvalues of any matrix that can be written as a linear combination of the matrices P_t or D_ℓ matrices. In particular, we will use the following estimate on the

eigenvalues of such matrices in our analysis of $\mathcal{M}$ from the previous section.

10. Lemma (Eigenvalue Estimates for Johnson Scheme). *Let* $Q = \sum_\ell \alpha_\ell D_\ell \in \mathcal{J}_{n,d}$ *and* $\beta_t = \sum_{t \leq \ell} \binom{t}{\ell} \alpha_\ell$ *for* $\alpha_\ell \geq 0$. *Then, for* $0 \leq j \leq r$,

$$\lambda_j(Q) \leq \sum_{t \geq j} \beta_t \binom{n-t-j}{d-t} \binom{d-j}{t-j}. \tag{8}$$

Proof.

$$\sum_\ell \alpha_\ell D_\ell = \sum_\ell \alpha_\ell \left(\sum_{t \geq \ell} (-1)^{t-\ell} \binom{t}{\ell} P_t \right)$$

$$= \sum_t P_t \left(\sum_{\ell \leq t} (-1)^{t-\ell} \binom{t}{\ell} \alpha_\ell \right) \tag{9}$$

$$\leq \sum_t P_t \left(\sum_{t \leq \ell} \alpha_\ell \right) \qquad = \sum_t \beta_t P_t$$

Using that Q and P_t share eigenspaces and applying Lemma Lemma 9, we obtain the estimate claimed. $\square$

PSDness of $\tilde{\mathbb{E}}$

11. Lemma (M is PSD). $\mathcal{M} \succeq 0$.

Proof. We are going to use Lemma Lemma 10. For any non-negative $\alpha_1, \alpha_2, \ldots, \alpha_d$, note that:

$$0 \preceq \sum_t \alpha_t P_t = \sum_{\ell=0}^{r} \left(\sum_{t=0}^{\ell} \alpha_t \right) \binom{\ell}{t} D_\ell \tag{10}$$

From the definition of $\mathcal{M}$, we know that $\mathcal{M} = \sum_{\ell=0}^{d} f(\ell) D_{2d-\ell}$. We will thus be done from above if we can find non-negative α_t such that $(\sum_{t=0}^{\ell} \alpha_t)\binom{\ell}{t} = f(2d-\ell)$ for every ℓ.

Observe that $f(2d-\ell) = f(2d) \cdot \dfrac{\binom{n-2d+\ell}{\ell}}{\binom{r-2d+\ell}{\ell}}$.

Choose $\alpha_t = \alpha_t = \dfrac{f(d)}{\binom{n}{d}} \cdot \dfrac{\binom{n-r}{t}}{\binom{r-2d+t-1}{t}}$.

We can now verify:

$$f(2d)\binom{n-2d+\ell}{\ell} = f(2d) \sum_{t=0}^{\ell} \binom{n-r}{t} \cdot \binom{r-2d+\ell-1}{\ell-t}$$

$$= \sum_{t=0}^{\ell} \alpha_t \cdot \binom{r-2d+t-1}{t}\binom{r-2d+\ell-1}{\ell-t} \tag{11}$$

$$= \sum_{t=0}^{\ell} \alpha_t \binom{\ell}{t} \cdot \binom{r-2d+\ell}{\ell}.$$

The lemma now follows and in fact shows that the minimum eigenvalue of $\mathcal{M}$ is α_d. $\qquad\square$

References

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