Digression to boosting, experts, dense models, and their quantum counterparts

There is a large collection of results across many fields such as:

- Duality in linear programming (Farkas [1902], Minkowski [1896])
- The Hahn-Banach theorem in functional analysis (Hahn [1927], Banach [1929])
- The minimax theorem in game theory (Neumann [1928])
- Regret minimization and expert learning (Hannan [1957], Littlestone and Warmuth [1989])
- Boosting in machine learning (Schapire [1990], Freund and Schapire [1995])
- The hard core lemma in computational complexity (Impagliazzo [1995])
- The dense model theorem in additive combinatorics (Green and Tao [2008], Tao and Ziegler [2008])

that all share the following characteristics:

- They appear initially counterintuitive
- They are incredibly useful
- They are not that hard to prove once you gather the nerve to conjecture that they could be true. In fact, they can all proven by some kind of a local search/improvements type of algorithm such as best response, multiplicative weights or gradient descent.

To show optimality of sos we will need to use a result in this framework, and specifically the generalization of such results into the quantum or positive semidefinite setting.

Regret minimization

Consider the following setting. There is some universe $U$ of assets. An investor strategy can be thought of as a distribution $\mu$ over the assets (which we can think of as either describing the way to partition the portfolio or as describing how to probabilistically sample a single asset to invest in). At each time period $t$, the investor comes up with a distribution $\mu_t$, the universe comes up with a function $f_t: U \rightarrow [-1, +1]$ and profit to the investor is $\mathbb{E}_{x \sim \mu_t} f_t(x)$. In the
setting of regret minimization (also known as expert learning) our goal is to come up with an investment strategy that would minimize the loss we suffer compared to the best fixed strategy in hindsight \( \mu^* \). That is, we wish to find a way such that if for \( t = 1, \ldots, T \) we compute \( \mu_t \) based on \( f_0, \ldots, f_{t-1} \) then we will minimize the maximum of
\[
\sum_{t=1}^{T} \mathbb{E}_{\mu^*} f_t - \sum_{t=1}^{T} \mathbb{E}_{\mu_t} f_t
\]
over all distributions \( \mu^* \) over \( U \).

The basic result in this area is the following:

**1. Theorem (Regret minimization).** For every parameter \( \eta \), and every choice of \( f_1, \ldots, f_t \) and distribution \( \mu^* \) we can choose \( \mu_t \) based only on \( f_1, \ldots, f_{t-1} \) such that
\[
\sum_{t=1}^{T} \mathbb{E}_{\mu^*} f_t \leq (1 + O(\eta)) \left[ \sum_{t=1}^{T} \mathbb{E}_{\mu_t} f_t \right] + \frac{1}{\eta} \Delta(\mu^* \| \mu_t) + O(1)
\]
where \( \Delta(\mu' \| \mu) \) denotes the KL divergence of \( \mu' \) from \( \mu \).

In particular if we set \( \mu_1 \) to be the uniform distribution, then since \( \Delta(\mu^* \| \mu_1) \leq \log |U| \) we can set \( \eta \) to be \( \sqrt{\log |U| / T} \) and get that the total regret is bounded by \( O(\sqrt{T \log |U|}) \) which (for \( T \gg \log |U| \)) is sublinear in \( T \).

**Proof.** We are going to simply let \( \mu_{t+1}(x) \) be equal to \( Z_t \mu_t(x) Z_t^{-1} f_t(x) \) where \( Z_t = \left( \mathbb{E}_{\mu_t} 2^{f_t(x)} \right)^{-1} \) is a normalization factor.

Now let us upper bound the decrease in distance between \( \mu^* \) and our current distribution by something related to the loss we suffer compared to the optimum:
\[
\Delta(\mu^* \| \mu_{t+1}) - \Delta(\mu^* \| \mu_t) = \mathbb{E}_{x \sim \mu^*} \log \left( \frac{\mu^*(x)}{\mu_{t+1}(x)} \right) - \mathbb{E}_{x \sim \mu^*} \left( \frac{\mu^*(x)}{\mu_t(x)} \right)
\]
which equals
\[
\mathbb{E}_{x \sim \mu^*} \left( \frac{\mu_t(x)}{\mu_{t+1}(x)} \right) = \mathbb{E}_{x \sim \mu^*} \log \left( \frac{1}{Z_t 2^{f_t(x)}} \right) = \log Z_t^{-1} - \eta \mathbb{E}_{\mu^*} f_t .
\]
but since \( Z_t^{-1} = \mathbb{E}_{\mu_t} 2^{f_t} \leq (\eta - O(\eta^2)) \mathbb{E}_{\mu_t} f_t \) we get
\[
\Delta(\mu^* \| \mu_{t+1}) - \Delta(\mu^* \| \mu_t) \leq \eta \left( (1 - \eta) \mathbb{E}_{\mu_t} f_t - \mathbb{E}_{\mu^*} f_t \right) .
\]

The telescopic sum of Eq. (5) over all \( t \) from 1 to \( T \) yields the theorem.
The proof of Theorem 1 actually yields more than just the statement. In particular the following two points are important:

• Our strategies \( \mu_1, \ldots, \mu_t \) are simple in the sense that they are composed of the initial prior \( \mu_1 \) reweighed by “few” of the functions \( f : U \to [0,1] \).

• The complexity of our strategy can be controlled by the KL distance from our prior to the optimal distribution. That is, if there were only few bits of information that we were missing, then there is a simple strategy that is nearly optimal.

This is a general (and very useful) phenomena that simple tests can be fooled by simple distributions (see, e.g., Trevisan et al. [2009]). In particular, the above proof establishes the following theorem:

2. Theorem (Simple tests can be fooled by simple distributions: classical version). Let \( \mathcal{F} \) be a collection of test functions mapping some universe \( U \) to \([-1, 1]\), and let \( \mu_{\text{opt}}, \mu_{\text{prior}} \) be some distribution over \( U \). Then there exists a distribution \( \mu \) such that

\[
\mathbb{E}_{\mu_{\text{opt}}} f - \mathbb{E}_{\mu_{\text{prior}}} f < \epsilon
\]

for every \( f \in \mathcal{F} \) and \( \mu \) is simple in the sense that it is obtained by reweighing \( \mu_{\text{prior}} \) using a function proportional to \( \epsilon \sum_{i=1}^{t} f_i \), where

\[
t = \Delta(\mu_{\text{opt}} \| \mu_{\text{prior}}) \text{poly}(1/\epsilon).
\]

Quantum version

We can extend the above observations to the quantum setting. Suppose that now the investor strategy is a quantum state \( \rho \) (i.e., psd matrix of trace 1) on a system with the universe of states \( U \), and the gain is now the probability that \( \rho \) passes some measurement which is a \( |U| \times |U| \) matrix \( M \) satisfying \( 0 \preceq M \preceq I \). The same algorithm works where we now use \( \rho_{t+1} \) as proportional to \( \rho_t e^{tM} \). This is known as the matrix multiplicative weights algorithm (e.g., see Arora et al. [2012]). A matrix exponential can be computed using the power series for the exponential (or by keeping the same eigenbasis and exponentiating the eigenvalues). The same analysis works except that we now replace the KL divergence of \( \mu_{\text{opt}} \) and \( \mu_{\text{prior}} \) by the corresponding quantum relative entropy which corresponds to the von Neumann entropy of the states. In particular we can get the following result:

\[
\mathbb{E}_{\mu_{\text{opt}}} f - \mathbb{E}_{\mu_{\text{prior}}} f < \epsilon
\]
3. Theorem (Simple tests can be fooled by simple distributions: classical version). Let $\mathcal{F}$ be a collection of quantum measurements over a system with universe of states $U$, where for every $f \in \mathcal{F}$, $-\Id_U \preceq f \preceq +\Id_U$ where $\Id_U$ denotes the $|U| \times |U|$ identity matrix and $\preceq$ denotes spectral domination. Let $\rho_{\text{opt}}, \rho_{\text{prior}}$ be two density matrices over $U$ (i.e., $|U| \times |U|$ psd matrices with trace 1). Then there exists a density $\rho$ such that

$$\text{Tr}(\rho f^*) - \text{Tr}(\rho_{\text{opt}} f^*) < \epsilon$$

(7)

for every $f \in \mathcal{F}$ and $\rho$ is simple in the sense that it is obtained by reweighing $\rho_{\text{prior}}$ using a function proportional to $\epsilon^{\sum_{i=1}^{f}}$ where $t = \Delta(\rho_{\text{opt}} \| \rho_{\text{prior}}) \text{poly}(1/\epsilon)$ and $\Delta(\rho \| \sigma)$ denotes the quantum relative entropy $\text{Tr}(\rho (\log \rho - \log \sigma))$.

References


